THE DEGREE OF MONOTONE APPROXIMATION

by

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THE DEGREE OF MONOTONE APPROXIMATION

§4.1 SUMMARY

Jackson type theorems are obtained for generalized monotone approximation. Let $E_{n,k}(f)$ be the degree of approximation of f by nth degree polynomials with kth derivative non-negative on $[-\frac{1}{4}, \frac{1}{4}]$. Then for each $k \geq 2$ there exists an absolute constant D_k, such that for all $f \in C[-\frac{1}{4}, \frac{1}{4}]$ with kth forward difference non-negative on $[-\frac{1}{4}, \frac{1}{4}]$; $E_{n,k}(f) \leq D_k\, \omega(f,n^{-1})$. If in addition $f' \in C[-\frac{1}{4}, \frac{1}{4}]$ then $E_{n,k}(f) \leq D_k n^{-1} \omega(f';n^{-1})$. Let $E^*_{n,2}(f)$ be the degree of approximation on $[-1,1]$, of f, by nth degree polynomials convex on the whole real line. Then there exists a constant M such that for each f convex on $[-1,1]$; $E^*_{n,2}(f) \leq M\omega(f,n^{-1})$. The results concerning $E_{n,k}$ are to appear in Beatson [1].

§4.2 INTRODUCTION

Let f be a function with non-negative kth forward difference over each set of k equally spaced points in $[-\frac{1}{4}, \frac{1}{4}]$ (equivalently any finite real interval). It is natural to ask whether Jackson type estimates hold for

$$E_{n,k}(f) = \inf_{\{p \in \Pi_n \,:\, p^{(k)}(x) \geq 0,\, x \in [-\frac{1}{4}, \frac{1}{4}]\}} \|f-p\|_{[\frac{1}{4}, \frac{1}{4}]} \quad ;$$

where the norm is the uniform norm, and Π_n is the space of algebraic polynomials of degree not exceeding n. In the case $k = 1$, Lorentz and Zeller [5] and Lorentz [6] have shown that there exists a constant D_1 such that if f is increasing on $[-\frac{1}{4}, \frac{1}{4}]$

$$E_{n,1}(f) \leqslant D_1 \omega(f,n^{-1}), \qquad n = 1,2,\ldots \quad , \tag{4.2.1}$$

where $\omega(f,.)$ denotes the modulus of continuity of f. If, in addition $f' \in C[\frac{1}{4}, \frac{1}{4}]$ then

$$E_{n,1}(f) \leqslant D_1 \, n^{-1} \, \omega(f',n^{-1}), \qquad n = 1,2,\ldots \quad . \tag{4.2.2}$$

Let f be a function convex on $[-1,1]$, and

$$E^*_{n,2}(f) = \inf_{\{p \in \Pi_n : p''(x) \geqslant 0, \ \forall x \in R\}} \|f-p\|_{[-1,1]} \ .$$

The lowest order Jackson type estimate will be shown for $E^*_{n,2}$. Higher order Jackson type estimates for $E^*_{n,2}$, if they exist, would have immediate practical application. Combined with standard arguments they would yield results concerning uniform approximation by reciprocals of polynomials on semi-infinite or infinite intervals.

§4.3 TWO JACKSON TYPE ESTIMATES OF $E_{n,k}$.

DeVore [3,4] has given a much simpler proof of the $k = 1$ results. Partly similar arguments, are used in this section, to show Jackson estimates analogous to (4.2.1), (4.2.2) for $E_{n,k}$.

Notation. Throughout $C_1, C_2, \ldots$ denote positive constants depending on k, but not depending on f, x or $n \geqslant k$. Whenever it causes no confusion, $\|.\|_\beta$ denotes $\|.\|_{[-\beta,\beta]}$ and $\omega(e,.)$ denotes $\omega_{[-\frac{1}{4}, \frac{1}{4}]}(e,.)$.

A function with non-negative kth difference on $[a,b]$ cannot, in general, be extended to a function with non-negative kth difference on a larger interval. For example the piecewise linear and convex function, $f \in C[0, \sum_{n=1}^{\infty} n^{-3}]$, with slope n on the interval $\left(\sum_{i=1}^{n-1} i^{-3}, \sum_{i=1}^{n} i^{-3}\right)$, cannot be extended to the right and remain convex. This motivates the construction of a pre-approximation (see Lemma 4.1) to f, to which we will apply appropriate polynomial convolution operators (see Lemma 4.2).

LEMMA 4.1. *Suppose* $k \geq 2$. *Let*

$$L_n(h,x) = (2\lambda)^{-k} \int_{-\lambda}^{\lambda} \cdots \int_{-\lambda}^{\lambda} h(x + t_1 + \ldots + t_k)dt_1 \ldots dt_k \quad (4.3.1)$$

where $h \in C[-\tfrac{1}{4}, \tfrac{1}{4}]$ *and*

$$\lambda = 1/8n, \qquad n = k, k+1, \ldots \quad . \quad (4.3.2)$$

Extend the definition of $L_n(h)$ *from*

$$[-\alpha,\alpha] = [-\tfrac{1}{4} + \frac{k}{8n}, \tfrac{1}{4} - \frac{k}{8n}] \quad (4.3.3)$$

to $[-\tfrac{1}{2}, \tfrac{1}{2}]$ *by adjoining, to the right and left, the Taylor polynomials of degree* k, *corresponding to* $L_n(h)$ *at the points* α, $-\alpha$. *Then there exists constants* E_k, F_k, G_k; $\bar{E}_k$, $\bar{F}_k$, $\bar{G}_k$; *such that; for all* $f \in C[-\tfrac{1}{4}, \tfrac{1}{4}]$ *with* $f(-\tfrac{1}{4}) = f(\tfrac{1}{4}) = 0$ *and non-negative* kth *difference on* $[-\tfrac{1}{4}, \tfrac{1}{4}]$; *for* $n = k, k+1, \ldots$;

$$L_n(f,x)^{(k)} \geq 0, \qquad x \in \mathbb{R} \quad , \quad (4.3.4)$$

$$\|L_n(f)^{(j)}\|_{\frac{1}{4}} \leq E_k\, n^j\, \omega(f,n^{-1}) \qquad (j = 1,\ldots, k-1), \quad (4.3.5)$$

$$\|L_n(f)^{(k)}\|_{\frac{1}{2}} \leq E_k\, n^k\, \omega(f,n^{-1}) \quad , \quad (4.3.6)$$

$$\|f - L_n(f)\|_{\frac{1}{4}} \leq F_k\, \omega(f,n^{-1}) \quad , \quad (4.3.7)$$

and

$$\|L_n(f)\|_{\frac{1}{4}} \leq G_k\, n\, \omega(f,n^{-1}) \quad . \quad (4.3.8)$$

If in addition $f' \in C[-\tfrac{1}{4}, \tfrac{1}{4}]$ *then*

$$\|L_n(f)^{(j)}\|_{\frac{1}{4}} \leq \bar{E}_k\, n^{j-1}\, \omega(f',n^{-1}) \qquad (j = 2,\ldots, k-1), \quad (4.3.5')$$

$$\|L_n(f)^{(k)}\|_{\frac{1}{2}} \leq \bar{E}_k\, n^{k-1}\, \omega(f',n^{-1}) \quad , \quad (4.3.6')$$

$$\|f - L_n(f)\|_{\frac{1}{4}} \leq \bar{F}_k\, n^{-1}\, \omega(f',n^{-1}) \quad , \quad (4.3.7')$$

and

$$\|L_n(f)^{(2-j)}\|_{\frac{1}{4}} \leq \bar{G}_k\, n^j \omega(f',n^{-1}) \qquad (j = 1,2). \quad (4.3.8')$$

Proof. For $x \in [-\alpha,\alpha]$

$$L_n(f,x) = (2\lambda)^{-k} \int_{-\lambda}^{\lambda} \cdots \int_{-\lambda}^{\lambda} \int_{x+t_2+\ldots+t_k-\lambda}^{x+t_2+\ldots+t_k+\lambda} f(\gamma)d\gamma\, dt_2 \ldots dt_k$$

implying

$$L_n(f,x)' = (2\lambda)^{-k} \int_{-\lambda}^{\lambda} \cdots \int_{-\lambda}^{\lambda} \Delta_{2\lambda}\, f(x + t_2 + \ldots + t_k - \lambda)dt_2 \ldots dt_k \ ;$$

repeating the argument, j times, $j = 1,\dots k$,

$$L_n(f,x)^{(j)} = (2\lambda)^{-k} \int_{-\lambda}^{\lambda} \cdot\cdot \int_{-\lambda}^{\lambda} \Delta_{2\lambda}^{j}\ f(x+t_{j+1}+\dots+t_k - j\lambda)\,dt_{j+1}\ \dots\ dt_k. \tag{4.3.9}$$

(4.3.4) follows immediately. (4.3.9) and the definition of λ imply

$$\|L_n(f)^{(j)}\|_\alpha \leqslant C_1\ n^j\ \omega(f,n^{-1}) \qquad (j = 1,\dots, k). \tag{4.3.10}$$

(4.3.5), (4.3.6) follow from (4.3.10) on estimating the derivatives of the Taylor polynomials extending $L_n(f)$ to the larger interval.

To prove (4.3.7). The definition of $L_n(f,x)$ clearly implies

$$\|f - L_n(f)\|_\alpha \leqslant C_2\ \omega(f,n^{-1}). \tag{4.3.11}$$

Also

$$\|f-L_n(f)\|_{[\alpha,\frac14]} \leqslant \|f-f(\alpha)\|_{[\alpha,\frac14]} + |f(\alpha)-L_n(f,\alpha)| + \|L_n(f,\alpha) - L_n(f)\|_{[\alpha,\frac14]};$$

so by (4.3.2); (4.3.11); (4.3.5), (4.3.6); and the manner in which $L_n(f)$ was extended

$$\|f - L_n(f)\|_{[\alpha,\frac14]} \leqslant C_3\ \omega(f,n^{-1}).$$

A similar result holds on $[-\frac14, -\alpha]$; (4.3.7) follows.

To prove (4.3.8). Note that (4.3.7) implies both

$$\omega(L_n(f),n^{-1}) \leqslant C_4\ \omega(f,n^{-1})$$

and

$$L_n(f,-\tfrac14) \leqslant F_k\ \omega(f,n^{-1}) \quad ;$$

the second since $f(-\frac14) = 0$; (4.3.8) follows.

We proceed to prove the results for $f' \in C[-\frac14, \frac14]$. Arguments analogous to those leading from (4.3.9) to (4.3.5), (4.3.6); lead from

$$L_n(f,x)^{(j)} = (2\lambda)^{-k} \int_{-\lambda}^{\lambda} \cdot\cdot \int_{-\lambda}^{\lambda} \Delta_{2\lambda}^{j-1}\ f'(x+t_j+\dots+t_k - (j-1)\lambda)\,dt_j\ \dots\ dt_k,$$

$(j = 1,\dots, k)$ to (4.3.5'), (4.3.6').

To show (4.3.7') use the quantitative Korovkin type estimate (see e.g. DeVore [3, pp28-32])

$$|L_n(f,x) - f(x)| \leqslant |f(x)||1 - L_n(1,x)| + |f'(x)||L_n((t-x),x)| + (1+\sqrt{L_n(1,x)})\alpha_n(x)\omega(f';\alpha_n(x)) \tag{4.3.12}$$

where

$$\alpha_n^2(x) = L_n((t-x)^2, x). \tag{4.3.13}$$

Now $\qquad \|1 - L_n(1)\| = \|L_n((t-x), x)\| = 0,$

while

$$L_n((t-x)^2,x) = (2\lambda)^{-k}\int_{-\lambda}^{\lambda}\cdots\int_{-\lambda}^{\lambda}(t_1+t_2+\cdots+t_k)^2\,dt_1\ldots dt_k$$
$$= k(2\lambda)^{-1}\int_{-\lambda}^{\lambda} t^2\,dt \leqslant c_5 n^{-2}.$$

Substituting into (4.3.12), (4.3.13) we find

$$\|L_n(f) - f\|_\alpha \leqslant c_6 n^{-1}\omega(f',n^{-1}). \tag{4.3.11'}$$

Since for this particular operator

$$L_n(f,x)' = L_n(f',x), \qquad x \in [-\alpha,\alpha]$$

and $L_n(f,x)'$ is continued outside $[-\alpha,\alpha]$ by adjoining the Taylor polynomials of degree $k-1$, corresponding to f', at either end point; reasoning, similar to that yielding (4.3.7), implies

$$\|f' - L_n(f)'\|_{\frac{1}{4}} \leqslant c_7\,\omega(f',n^{-1}). \tag{4.3.14}$$

Writing

$$\|f - L_n(f)\|_{[\alpha,\frac{1}{4}]} \leqslant |f(\alpha) - L_n(f,\alpha)| + \int_\alpha^{\frac{1}{4}}|f'(t) - L_n(f,t)'|dt\ ;$$

(4.3.11'); (4.3.2) and (4.3.14) imply

$$\|f - L_n(f)\|_{[\alpha,\frac{1}{4}]} \leqslant c_8 n^{-1}\omega(f',n^{-1}).$$

Combining the above, the similar result on $[-\frac{1}{4}, -\alpha]$, and (4.3.11') proves (4.3.7').

To show (4.3.8'). Note (4.3.14) implies

$$\omega(L_n(f)', n^{-1}) \leqslant c_9\,\omega(f',n^{-1})$$

and also

$$|L_n(f,\xi)'| \leqslant c_7\,\omega(f',n^{-1}) \text{ where } f'(\xi) = 0,\ -\tfrac{1}{4} < \xi < \tfrac{1}{4};$$

the existence of such an ξ following from $f(-\frac{1}{4}) = f(\frac{1}{4}) = 0$.

Hence

$$\|L_n(f)'\|_{\frac{1}{4}} \leqslant C_{10}\ n\ \omega(f',n^{-1}).$$

(4.3.8') follows since (4.3.7') implies

$$|L_n(f,-\tfrac{1}{4})| \leqslant \bar{F}_k\ n^{-1}\ \omega(f',n^{-1}). \qquad //$$

We now know how well $L_n(f)$ approximates f, and concern ourselves with how well $L_n(f)$ may be approximated by convolutions with positive polynomials.

<u>LEMMA 4.2</u>. *Suppose* $k \geqslant 2$. *Then there exist constants* H_k, I_k *and a sequence of even positive algebraic polynomials* $\{\lambda_n\}_{n=k}^{\infty}$ *satisfying*

$$\int_{-1}^{1} \lambda_n(t)\,dt = 1 \quad , \tag{4.3.15}$$

and

$$\|\lambda_n^{(j)}\|_{[-1,1]\setminus[-\frac{1}{4},\frac{1}{4}]} \leqslant H_k\ n^{2-4k+2j}\ (\leqslant H_k\ n^{-2k}), \qquad (j = 0,\ldots, k-1). \tag{4.3.16}$$

Further if f *satisfies the conditions of Lemma* 4.1, $g = L_n(f)$ *and*

$$L_n^*(g) = \int_{-\frac{1}{2}}^{\frac{1}{2}} g(t)\ \lambda_n(t-x)\,dt \quad ; \tag{4.3.17}$$

then if $f \in C[-\frac{1}{4}, \frac{1}{4}]$

$$\|g - L_n^*(g)\|_{\frac{1}{4}} \leqslant I_k\ \omega(f,n^{-1}) \quad ; \tag{4.3.18}$$

and if $f' \in C[-\frac{1}{4}, \frac{1}{4}]$

$$\|g - L_n^*(g)\|_{\frac{1}{4}} \leqslant I_k\ n^{-1}\ \omega(f',n^{-1}). \tag{4.3.19}$$

<u>*Proof*</u>. Let $\lambda_k = \lambda_{k+1} = \ldots = \lambda_{4k-1} \equiv \frac{1}{2}$.

For $n \geqslant 2k$, let

$$\lambda_{4n-4k}(t) = c_n[P_{2n}(t)/((t^2 - x_{1,2n}^2)\ldots(t^2 - x_{k,2n}^2))]^2 \quad , \tag{4.3.20}$$

where P_{2n} is the Legendre polynomial of degree 2n and $x_{1,2n},\ldots,x_{n,2n}$ are its positive zeros in increasing order. c_n is a normalizing constant for (4.3.15). Define the remaining λ_n's with the relation

$$\lambda_{4n+1} = \lambda_{4n+2} = \lambda_{4n+3} = \lambda_{4n} \quad , \quad n \geqslant k.$$

Observe firstly that a theorem of Bruns (see e.g. DeVore [3, p.20]) implies

$$C_{11}\, n^{-1} \leqslant x_{1,2n} < \ldots < x_{k,2n} \leqslant C_{12}\, n^{-1}, \quad n > k \tag{4.3.21}$$

Using the normalization $\|P_n\|_{[-1,1]} = 1$ and the corresponding Taylor expansion of P_n (see e.g. Davis [2, p.365]),

$$|P_{2n}(0)| = 2^{-2n} \binom{2n}{n} = (1+o(1))/\sqrt{\pi n} \quad , \tag{4.3.22}$$

the last equality being a consequence of Stirling's formula.

(4.3.20), (4.3.21) and (4.3.22) together imply

$$\lambda_{4n-4k}(0) \geqslant C_{13}\, c_n\, n^{4k-1}, \quad n \geqslant 2k. \tag{4.3.23}$$

Let $n \geqslant 2k$. Write

$$1 = \int_{-1}^{1} \lambda_{4n-4k}(t)\,dt = \sum_{k=-n}^{n} A_k(2n+1)\lambda_{4n-4k}(x_{k,2n+1}) \;;$$

where the $A_k(2n+1)$ are the weights of the Gaussian quadrature formula, exact for polynomials of degree 4n+1, with nodes at the zeros of the Legendre polynomial of degree 2n+1. Therefore

$$1 \geqslant A_0(2n+1)\lambda_{4n-4k}(0)$$

and since (Szego [8, p.350]), $A_0\,(2n+1) = \frac{\pi}{2n+1}\,(1 + o(1))$

$$\lambda_{4n-4k}(0) \leqslant C_{14}\, n. \tag{4.3.24}$$

(4.3.23) and (4.3.24) imply

$$c_n \leqslant C_{15}\, n^{2-4k} \quad ;$$

which together with the normalization of the P_n, the definition of the λ_n, and (4.3.21) implies

$$\|\lambda_n\|_{[-1,1]\backslash[-\frac{1}{4},\frac{1}{4}]} \leqslant C_{16}\, n^{2-4k} \quad .$$

(4.3.16) follows by means of Markov's inequality.

It remains to show the order of approximation results. We cannot use the standard quantitative Korovkin theorem as $\omega_{[-\frac{1}{2},\frac{1}{2}]}(g,n^{-1}) \neq O(\omega_{[-\frac{1}{2},\frac{1}{2}]}(f,n^{-1}))$; at least not in general. However a related method is applicable.

Again let $n \geqslant 2k$. The polynomial $t^{2k}\lambda_{4n-4k}(t)$ is of degree $4n-2k$. Therefore for $j = 1,\dots, k$

$$M_j = \int_{-1}^{1} t^{2j}\lambda_{4n-4k}(t)\,dt = 2\sum_{i=1}^{n} x_{i,2n}^{2j} A_i(2n)\lambda_{4n-4k}(x_{i,2n}) \quad ;$$

where the $A_i(2n)$ are the weights of the Gaussian quadrature formula, exact for polynomials of degree $4n-1$, with nodes at the zeros of the Legendre polynomial of degree $2n$. Since λ_{4n-4k} has zeros at $x_{k+1,2n},\dots, x_{n,2n}$,

$$M_j = 2\sum_{i=1}^{k} x_{i,2n}^{2j} A_i(2n)\lambda_{4n-4k}(x_{i,2n}).$$

Since also λ_{4n-4k} has a local maximum on $[-x_{k+1,2n}, x_{k+1,2n}]$ at zero, and (Szego [8, p.350])

$$A_i(2n) \leqslant \frac{\pi}{2n}(1+o(1)) \qquad (i = 1,\dots, k) \quad ,$$

(4.3.21), (4.3.24) imply

$$\int_{-1}^{1} t^{2j}\lambda_n(t)\,dt \leqslant C_{17}\, n^{-2j}, \quad j = 1, \quad ,k, \quad n \geqslant k. \tag{4.3.25}$$

(4.3.25), (4.3.16) and that $\lambda_n(t)$ is even and non-negative may be used to estimate certain quantities involving L_n^*. All the estimates are uniform in $|x| \leqslant \frac{1}{4}$.

$$0 \leqslant 1-L_n^*(1,x) = \int_{-1}^{1}\lambda_n(t)\,dt - \int_{-\frac{1}{2}-x}^{\frac{1}{2}-x}\lambda_n(t)\,dt \leqslant 2\int_{\frac{1}{4}}^{1}\lambda_n(t)\,dt \leqslant C_{18}\, n^{2-4k}. \tag{4.3.26}$$

$$\begin{aligned} 0 \leqslant L_n^*((t-x)^{2j}, x) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} (t-x)^{2j}\lambda_n(t-x)\,dt \\ &= \int_{-\frac{1}{2}-x}^{\frac{1}{2}-x} t^{2j}\lambda_n(t)\,dt \\ &\leqslant \int_{-1}^{1} t^{2j}\lambda_n(t)\,dt \end{aligned}$$

and applying (4.3.25)

$$o \leqslant L_n^*((t-x)^{2j},x) \leqslant C_{19}\, n^{-2j}, \quad j = 1,\ldots, k. \tag{4.3.27}$$

$$\begin{aligned} o \leqslant L_n^*(|t-x|^k, x) &\leqslant \int_{-1}^{1} |t|^k \lambda_n(t)\,dt \\ &\leqslant \left(\int_{-1}^{1} t^{2k} \lambda_n(t)\,dt\right)^{\frac{1}{2}} \\ &\leqslant C_{20}\, n^{-k} , \end{aligned} \tag{4.3.28}$$

where we have used the Schwartz inequality, (4.3.15) and (4.3.27).

For j odd,

$$\begin{aligned} |L_n^*((t-x)^j,x)| &= \left|\int_{-\frac{1}{2}-x}^{\frac{1}{2}-x} t^j \lambda_n(t)\,dt\right| \\ &\leqslant 2\int_{\frac{1}{4}}^{1} t^j \lambda_n(t)\,dt \end{aligned}$$

since λ_n is even. Applying (4.3.16)

$$|L_n^*((t-x)^j,x)| \leqslant C_{21}\, n^{2-4k}, \quad j = 1,3,5,\ldots \quad . \tag{4.3.29}$$

If $t\in[-\frac{1}{2},\frac{1}{2}]$ and $x\in[-\frac{1}{4},\frac{1}{4}]$, Taylor's theorem gives

$$g(t) = \left(\sum_{j=0}^{k-1} \frac{g^{(j)}(x)(t-x)^j}{j!}\right) + \frac{1}{(k-1)!}\int_x^t g^{(k)}(u)(t-u)^{k-1}du. \tag{4.3.30}$$

Since the last term on the right hand side is bounded in modulus by $\frac{1}{k!}|t-x|^k \|g^{(k)}\|_{[-\frac{1}{2},\frac{1}{2}]}$, the linearity and monotonicity of L_n^* imply

$$\left|L_n^*(g,x) - \sum_{j=0}^{k-1} \frac{g^{(j)}(x)}{j!} L_n^*((t-x)^j,x)\right| \leqslant \frac{1}{k!}\|g^{(k)}\|_{[-\frac{1}{2},\frac{1}{2}]} L_n^*(|t-x|^k,x);$$

or

$$\begin{aligned} |L_n^*(g,x) - g(x)| \leqslant |g(x)|\,|1 - L_n^*(1,x)| &+ \sum_{j=1}^{k-1} \frac{|g^{(j)}(x)|}{j!}|L_n^*((t-x)^j,x)| \\ &+ \frac{1}{k!}\|g^{(k)}\|_{[-\frac{1}{2},\frac{1}{2}]} L_n^*(|t-x|^k,x). \end{aligned}$$

Thus

$$\begin{aligned} \|L_n^*(g,x) - g(x)\|_{[-\frac{1}{4},\frac{1}{4}]} \leqslant \|g\|_{[-\frac{1}{4},\frac{1}{4}]}\|1-L_n^*(1)\|_{[-\frac{1}{4},\frac{1}{4}]} & \\ + \sum_{j=1}^{k-1} \frac{\|g^{(j)}\|_{[-\frac{1}{4},\frac{1}{4}]}}{j!}\|L_n^*((t-x)^j,x)\|_{[-\frac{1}{4},\frac{1}{4}]} & \\ + \frac{1}{k!}\|g^{(k)}\|_{[-\frac{1}{2},\frac{1}{2}]}\|L_n^*(|t-x|^k,x)\|_{[-\frac{1}{4},\frac{1}{4}]} & . \end{aligned}$$

Combining the above, the estimates of all the terms involving g from Lemma 4.1 ($g = L_n(f)$), and the estimates (4.3.26), (4.3.27), (4.3.28), (4.3.29) of all the $\|L_n^*(.,.)\|$'s yields (4.3.18), (4.3.19). //

Given Lemmas 4.1 and 4.2 it remains to discuss how close $L_n^*(g)$ is to a polynomial with non-negative kth derivative on $[-\frac{1}{4},\frac{1}{4}]$.

<u>THEOREM 4.3.</u> *For each* $k \geqslant 2$ *there exists a constant* D_k, *such that for all* $h \in C[-\frac{1}{4},\frac{1}{4}]$ *with kth forward difference non-negative on* $[-\frac{1}{4},\frac{1}{4}]$

$$E_{n,k}(h) \leqslant D_k\, \omega_{[-\frac{1}{4},\frac{1}{4}]}(h,n^{-1}), \qquad n = k,k+1,\ldots \quad .$$

If in addition $h' \in C[-\frac{1}{4},\frac{1}{4}]$ *then*

$$E_{n,k}(h) \leqslant D_k\, n^{-1}\omega_{[-\frac{1}{4},\frac{1}{4}]}(h',n^{-1}), \qquad n = k,k+1,\ldots \quad .$$

<u>*Proof.*</u> Fix $k \geqslant 2$. Let $f = h - \rho$ where

$$\rho(x) = h(-\tfrac{1}{4}) + 2(h(\tfrac{1}{4}) - h(-\tfrac{1}{4}))(x+\tfrac{1}{4}).$$

Clearly $\omega(f,n^{-1}) \leqslant 2\omega(h,n^{-1})$ and when h' exists $\omega(f',n^{-1}) = \omega(h',n^{-1})$. Lemmas 4.1, and 4.2 apply to f. Writing

$$\bar{L}_n(h) = \rho(x) + L_n^*(L_n(f))$$

Lemmas 4.1 and 4.2 imply

$$\begin{aligned}\|h - \bar{L}_n(h)\|_{\frac{1}{4}} &= \|f - L_n^*(L_n(f))\| \\ &\leqslant \|f - L_n(f)\|_{\frac{1}{4}} + \|L_n(f) - L_n^*(L_n(f))\|_{\frac{1}{4}} \\ &\leqslant \begin{cases} C_{22}\ \omega(h,n^{-1}), & h \in C[-\frac{1}{4},\frac{1}{4}], \\ C_{22}\ n^{-1}\ \omega(h',n^{-1}), & h' \in C[-\frac{1}{4},\frac{1}{4}]. \end{cases}\end{aligned} \tag{4.3.31}$$

Let $g = L_n(f)$. Then

$$\bar{L}_n(h) = \rho(x) + L_n^*(g) = \rho(x) + \int_{-\frac{1}{2}}^{\frac{1}{2}} g(t)\lambda_n(t-x)\,dt,$$

$$\begin{aligned}\bar{L}_n(h,x)' &= \rho'(x) + \int_{-\frac{1}{2}}^{\frac{1}{2}} g(t)\,.\, -\lambda_n'(t-x)\,dt \\ &= \rho'(x) + [-g(t)\lambda_n(t-x)]_{-\frac{1}{2}}^{\frac{1}{2}} + \int_{-\frac{1}{2}}^{\frac{1}{2}} g'(t)\lambda_n(t-x)\,dt.\end{aligned}$$

$k \geqslant 2$ alternate differentiations and integrations by parts yield;

$$\bar{L}_n(h,x)^{(k)} = (-1)^k \left[\sum_{j=0}^{k-1} (-1)^j \left[g^{(j)}(t)\lambda_n^{(k-1-j)}(t-x)\right]_{t=-\frac12}^{t=\frac12}\right] + \int_{-\frac12}^{\frac12} g^{(k)}(t)\lambda_n(t-x)\,dt$$
$$= r(x) + \int_{-\frac12}^{\frac12} g^{(k)}(t)\lambda_n(t-x)\,dt.$$

(4.3.4) and the positivity of the kernels imply the second term on the right hand side is non-negative. The estimates (4.3.5), (4.3.8); (4.3.16) imply

$$\|r\|_{\frac14} \leqslant C_{23}\, n^{-2k+1}\omega(h,n^{-1}), \qquad h \in C[-\tfrac14, \tfrac14],$$

and the estimates (4.3.5'), (4.3.8'); (4.3.16) imply

$$\|r\|_{\frac14} \leqslant C_{24}\, n^{-2k+2}\omega(h', n^{-1}), \qquad h' \in C[-\tfrac14, \tfrac14]$$

In the first case let

$$p_n(x) = \bar{L}_n(h,x) + \frac{x^k}{k!} C_{23}\, n^{-2k+1}\omega(h, n^{-1}),$$

and in the second let

$$p_n(x) = \bar{L}_n(h,x) + \frac{x^k}{k!} C_{24}\, n^{-2k+2}\omega(h', n^{-1}).$$

Then $p_n^{(k)}(x)$ is non-negative on $[-\frac14,\frac14]$; and by (4.3.31) $p_n(x)$ provides the estimate of the theorem.

§4.4 A JACKSON TYPE ESTIMATE OF $E^*_{n,2}$.

The argument, used in this section, is derived from the delightful proof of Jackson's theorem given by Passow [7]. Define $E^*_{n,2}(f)$ as in section 4.2.

THEOREM 4.4. *There exists a constant* M, *such that for any function* f *convex on* $[-1,1]$

$$E^*_{n,2} \leqslant M\omega_{[-1,1]}(f,n^{-1}), \qquad n = 2,3,4,\ldots \qquad .$$

Proof. Following [7] construct the polygonal pre-approximation $L(x)$ with : $L(k/n) = f(k/n)$, $k = -n,\ldots, n$; and L linear in each of the intervals $[k/n\ ,\ (k+1)/n]$. Then $|L(x) - f(x)| \leqslant \omega_{[-1,1]}(f,n^{-1})$ for all

$x \in [-1,1]$ and L is convex with f. Let S_k be the slope of L(x) in $((k-1)/n,\ k/n)$ and let

$$a_k = (-S_k + S_{k+1})/2\ , \qquad -n+1 \leqslant k \leqslant n-1 \quad ;$$

$$a_n = -(S_n + S_{-n+1})/2 = -a_{-n} \quad .$$

Then

$$L(x) = A_1 + \sum_{k=-n+1}^{n} a_k |x - k/n| = A_1 + \int_{-1}^{1} |x-t| dg_1(t) \quad ; \qquad (4.4.1)$$

where $g_1(t)$ is the step function having jumps at $x = k/n$ $(k = -n+1,\ldots,n)$ equal to a_k, $g(-1) = 0$, and A_1 a constant. Alternatively L(x) may be expanded as

$$L(x) = A_2 + \sum_{k=-n}^{n-1} a_k \left|x - \frac{k}{n}\right| = A_2 + \int_{-1}^{1} |x-t| dg_2(t) \quad ; \qquad (4.4.2)$$

where $g_2(t)$ is the step function having jumps at $x = \frac{k}{n}$ $(k = -n+1,\ldots,n-1)$ equal to a_k, $g(-1) = 0$, $g(x) = a_{-n}$ for $-1 < x < -1 + (1/n)$; and A_2 is a constant. These expansions are easily verified by calculating the slope of $\Sigma a_k |x - k/n|$ in each subinterval $((k-1)/n,\ k/n)$.

Since the slope of L is increasing, a_i will be non-negative for $i = -n+1,\ldots,n-1$. Also a_n is either negative or non-negative; hence at least one of g_1, g_2 will be increasing. Let

$$L(x) = A + \int_{-1}^{1} |x - t| dg(t) \qquad (4.4.3)$$

be an expression (4.4.1), (4.4.2) with g_j increasing. Then

<u>LEMMA 4.5</u>. *Let* q(x) *be a polynomial of degree not exceeding* n, *convex on the whole real line, satisfying* q(0) = 0 and

$$\int_{-2}^{2} |d\{|x| - q(x)\}| \leqslant b/n.$$

Then

$$Q_n(x) = A + \int_{-1}^{1} q(x-t)\, dg(t)$$

is a polynomial of degree not exceeding n, *convex on the whole real line, satisfying*

$$\max_{-1\leqslant x\leqslant 1} |f(x) - Q_n(x)| \leqslant (2b+1)\ \omega_{[-1,1]}(f,n^{-1}).$$

Proof. The degree of the approximation follows exactly as in [7, Lemma 1]. The argument is repeated for completeness.

$$\begin{aligned}\left|f(x) - A - \int_{-1}^{1} q(x-t)\,dg(t)\right| &\leqslant \left|f(x) - A - \int_{-1}^{1} |x-t|\,dg(t)\right| \\ &\quad + \left|\int_{-1}^{1} \{|x-t| - q(x-t)\}dg(t)\right| \\ &\leqslant \omega_{[-1,1]}(f,n^{-1}) + \Big|\{|x-t| - q(x-t)\}g(t)\Big|_{-1}^{1} \\ &\quad - \int_{-1}^{1} g(t)\ d\{|x-t| - q(x-t)\}\Big| \\ &\leqslant \omega_{[-1,1]}(f,n^{-1}) + |g(1)|\frac{b}{n} + \max_{-1\leqslant t\leqslant 1} |g(t)|b/n.\end{aligned}$$

Now

$$\max_{-1\leqslant x\leqslant 1}|g(t)| = \left|\begin{array}{lll} \displaystyle\max_{-n+1\leqslant j\leqslant n} \left|\sum_{k=-n+1}^{j} a_k\right| & , & \text{if } g = g_1\ , \\ \displaystyle\max_{-n\leqslant j\leqslant n-1} \left|\sum_{k=-n}^{j} a_k\right| & , & \text{if } g = g_2\ , \end{array}\right.$$

$$\leqslant \max_j |s_j| \leqslant n\omega_{[-1,1]}(f,n^{-1}).$$

Thus

$$\left|f(x) - A - \int_{-1}^{1} q(x-t)\,dg(t)\right| \leqslant (2b+1)\omega_{[-1,1]}(f,n^{-1}).$$

The convexity of Q_n follows from the convexity of q and the monotonicity of g, since

$$Q_n''(x) = \int_{-1}^{1} q''(x-t)\,dg(t). \qquad //$$

LEMMA 4.6. *There exists a constant* $c > 0$, *and for each* $n = 1,2,3,\ldots$ *a polynomial* q_{4n-2} *of degree* $4n-2$, *convex on the whole real line satisfying* $q_{4n-2}(0) = 0$ *and*

$$\int_{-2}^{2} |d\{|x| - q_{4n-2}(x)\}| \leqslant c/n. \qquad (4.4.4)$$

Proof. Let

$$\lambda_{4n-4}(t) = c_n(P_{2n}(t)/(t^2 - x_{1,2n}^2))^2$$

where P_{2n} is the Legendre polynomial of degree 2n ; $x_{1,2n}$ its smallest positive zero; and c_n is a normalizing constant chosen so that

$$\int_{-1}^{1} \lambda_{4n-4}(t)\,dt = 1. \qquad (4.4.5)$$

Then [3, pp174-176] λ_{4n-4} is an even, non-negative, algebraic polynomial of degree $4n-4$ such that

$$0 < \int_{-1}^{1} \lambda_{4n-4}(t)\, t^2\, dt \leqslant C_1/n^2 \qquad (4.4.6)$$

for some constant C_1, $n = 1,2,\ldots$. (4.4.5), (4.4.6) and the Schwartz inequality imply

$$0 < \int_{-1}^{1} |t|\lambda_{4n-4}(t)\,dt \leqslant C_2/n. \qquad (4.4.7)$$

Take as the approximation to $|x|$

$$q_{4n-2}(x) = \int_0^x \left(\int_0^u \lambda_{4n-4}(t/2)\,dt\right) du\,. \qquad (4.4.8)$$

The non negativity of λ_{4n-4} implies the convexity of q_{4n-2}. Also ((4.4.8))

$$q_{4n-2}(0) = q'_{4n-2}(0) = 0\ ; \qquad (4.4.9)$$

using in addition the properties of λ_{4n-4}

$$\begin{aligned} -1 = q'_{4n-2}(-2) \leqslant q'_{4n-2}(x) \leqslant 0\,, &\qquad -2 \leqslant x \leqslant 0, \\ 0 \leqslant q'_{4n-2}(x) \leqslant q'_{4n-2}(2) = 1\,, &\qquad 0 \leqslant x \leqslant 2\,. \end{aligned} \qquad (4.4.10)$$

From (4.4.9), (4.4.10) and the eveness of q_{4n-2}, it follows that ; $|x| - q_{4n-2}(x)$ is monotone decreasing on $[-2,0]$, monotone increasing on $[0,2]$, with

$$\int_{-2}^{2} |d\{|x| - q_{4n-2}(x)\}| = 2(2 - q_{4n-2}(2))\ . \qquad (4.4.11)$$

Taylor's theorem implies

$$
\begin{aligned}
q_{4n-2}(x) &= q_{4n-2}(0) + xq'_{4n-2}(0) + \int_0^x q''_{4n-2}(u)(x-u)\,du \\
&= \int_0^x \lambda_{4n-4}(u/2)(x-u)\,du .
\end{aligned}
$$

Hence

$$
\begin{aligned}
2 - q_{4n-2}(2) &= 2 - \int_0^2 \lambda_{4n-4}(u/2)(2-u)\,du \\
&= 2 - 4\int_0^1 \lambda_{4n-4}(t)(1-|t|)\,dt \\
&= 2\left(1 - \int_{-1}^1 \lambda_{4n-4}(t)(1-|t|)\,dt\right) \\
&= 2\int_{-1}^1 \lambda_{4n-4}(t)|t|\,dt ,
\end{aligned}
$$

by the eveness of λ_{4n-4} and (4.4.5). Now (4.4.7) implies
$0 < 2 - q_{4n-2}(2) \leqslant 2C_2/n$.
Substitute in (4.4.11) to obtain ;

$$
\int_{-2}^2 |d\{|x| - q_{4n-2}(x)\}| \leqslant C/n ,
$$

where $C = 4C_2$ does not depend on n; as required. //

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